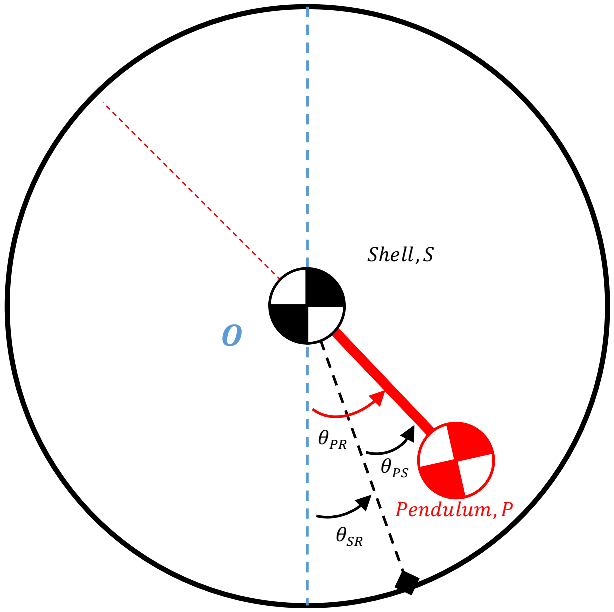

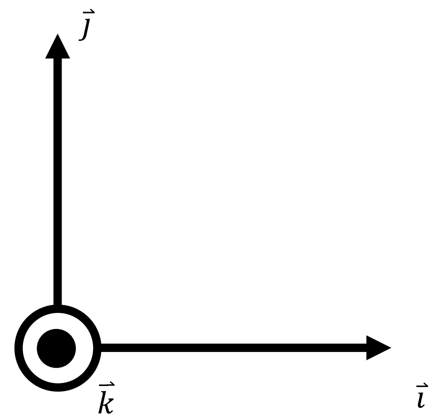

Degrees of Freedom → 2

Reference-Moving Frame Relationship:

$$\dot{\theta}_{MR} = \dot{\theta}_{PR} = \dot{\theta}_{PS} + \dot{\theta}_{SR}$$

Assuming no-slip conditions,

$$v_{SR} = -r_s\dot{\theta}_{SR}$$
$$v_{PR} = v_{SR} + r_P\dot{\theta}_{PR} = -r_s\dot{\theta}_{SR} + r_P\dot{\theta}_{PR}$$

Total Translational Kinetic Energy,

$$E_{K,T} = \frac{1}{2}m_S {v_{SR}}^2 + \frac{1}{2}m_P {v_{PR}}^2$$

$$E_{K,T} = \frac{1}{2}m_S {r_S}^2 {\dot{\theta}_{SR}}^2 + \frac{1}{2}m_P\left(-r_S\dot{\theta}_{SR} + r_P\dot{\theta}_{PR}\right)^2$$

$$E_{K,T} = \frac{1}{2}m_S {r_S}^2 {\dot{\theta}_{SR}}^2 + \left(\frac{1}{2}m_P {r_s}^2 {\dot{\theta}_{SR}}^2 + \frac{1}{2}m_P {r_P}^2 {\dot{\theta}_{PR}}^2 - m_P r_s r_P \dot{\theta}_{SR}\dot{\theta}_{PR}\right)$$

Total Rotational Kinetic Energy,

$$E_{R,T} = \frac{1}{2}I_S {\dot{\theta}_{SR}}^2 + \frac{1}{2}I_P {\dot{\theta}_{PR}}^2$$

$$E_{R,T} = \frac{1}{2}I_S {\dot{\theta}_{SR}}^2 + \frac{1}{2}I_P {\dot{\theta}_{PR}}^2$$

Total Potential Energy,

$$U = -m_P g r_P \cos\theta_{PR}$$

Non-conservative Work from Input Torque,

$$\delta W = \tau\delta\theta_{PS} = \tau\delta\theta_{PR} - \tau\delta\theta_{SR}$$

Non-conservative Work due to Damping from the Gearing Mechanisms,

$$\delta W = -br\dot{\theta}_{PS}(\delta r\theta_{PS}) = -br^2\dot{\theta}_{PS}\delta\theta_{PS} = -br^2\dot{\theta}_{PR}\delta\theta_{PR} + br^2\dot{\theta}_{SR}\delta\theta_{SR}$$

Using Lagrange Method,

$$[(m_S + m_P)r_S^2 + I_S]\ddot{\theta}_{SR} - m_P r_s r_P \ddot{\theta}_{PR} = -\tau + br^2\dot{\theta}_{SR}$$
$$[m_P r_P^2 + I_P]\ddot{\theta}_{PR} - m_P r_s r_P \ddot{\theta}_{SR} + m_P g r_P \sin\theta_{PR} = \tau - br^2\dot{\theta}_{PR}$$

Simplifying, we get,

(1),

$$\ddot{\theta}_{PR} = \frac{[(m_S + m_P)r_S^2 + I_S]}{m_P r_s r_P}\ddot{\theta}_{SR} - \frac{br^2}{m_P r_s r_P}\dot{\theta}_{SR} + \frac{\tau}{m_P r_s r_P}$$
$$\ddot{\theta}_{PR} - \frac{m_P r_s r_P}{[m_P r_P^2 + I_P]}\ddot{\theta}_{SR} + \frac{m_P g r_P}{[m_P r_P^2 + I_P]}\sin\theta_{PR} = \frac{\tau - br^2\dot{\theta}_{PR}}{[m_P r_P^2 + I_P]}$$

or (2),

$$\ddot{\theta}_{SR} = \frac{m_P r_s r_P}{[(m_S + m_P)r_S^2 + I_S]}\ddot{\theta}_{PR} + \frac{br^2}{[(m_S + m_P)r_S^2 + I_S]}\dot{\theta}_{SR} - \frac{\tau}{[(m_S + m_P)r_S^2 + I_S]}$$
$$\frac{[m_P r_P^2 + I_P]}{m_P r_s r_P}\ddot{\theta}_{PR} - \ddot{\theta}_{SR} + \frac{m_P g r_P}{m_P r_s r_P}\sin\theta_{PR} = \frac{\tau - br^2\dot{\theta}_{PR}}{m_P r_s r_P}$$

(1) and (2) give rise to,

$$\left\{\frac{[(m_S+m_P)r_S^2+I_S]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[m_P r_P^2+I_P]}\right\}\ddot{\theta}_{SR} - \frac{br^2}{m_P r_s r_P}\dot{\theta}_{SR} + \frac{m_P g r_P}{[m_P r_P^2+I_P]}\sin\theta_{PR} + \frac{br^2}{[m_P r_P^2+I_P]}\dot{\theta}_{PR} = \frac{\tau}{[m_P r_P^2+I_P]} - \frac{\tau}{m_P r_s r_P}$$
$$\left\{\frac{[m_P r_P^2+I_P]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[(m_S+m_P)r_S^2+I_S]}\right\}\ddot{\theta}_{PR} + \frac{m_P g r_P}{m_P r_s r_P}\sin\theta_{PR} + \frac{br^2}{m_P r_s r_P}\dot{\theta}_{PR} - \frac{br^2}{[(m_S+m_P)r_S^2+I_S]}\dot{\theta}_{SR} = \frac{\tau}{m_P r_s r_P} - \frac{\tau}{[(m_S+m_P)r_S^2+I_S]}$$

Converting to State-Space,

$$z_1 = \theta_{SR}$$
$$z_2 = \theta_{PR}$$
$$z_3 = \dot{z_1} = \dot{\theta}_{SR}$$
$$z_4 = \dot{z_2} = \dot{\theta}_{PR}$$

$$\dot{z_3} = \frac{\frac{br^2}{m_P r_s r_P}}{\left\{\frac{[(m_S+m_P)r_S^2+I_S]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[m_P r_P^2+I_P]}\right\}} z_3 - \frac{\frac{m_P g r_P}{[m_P r_P^2+I_P]}}{\left\{\frac{[(m_S+m_P)r_S^2+I_S]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[m_P r_P^2+I_P]}\right\}}\sin z_2$$
$$- \frac{\frac{br^2}{[m_P r_P^2+I_P]}}{\left\{\frac{[(m_S+m_P)r_S^2+I_S]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[m_P r_P^2+I_P]}\right\}} z_4 + \frac{\frac{\tau}{[m_P r_P^2+I_P]}}{\left\{\frac{[(m_S+m_P)r_S^2+I_S]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[m_P r_P^2+I_P]}\right\}}$$
$$- \frac{\frac{\tau}{m_P r_s r_P}}{\left\{\frac{[(m_S+m_P)r_S^2+I_S]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[m_P r_P^2+I_P]}\right\}}$$

$$\dot{z_4} = \frac{\frac{br^2}{[(m_S+m_P)r_S^2+I_S]}}{\left\{\frac{[m_P r_P^2+I_P]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[(m_S+m_P)r_S^2+I_S]}\right\}} z_3 - \frac{\frac{m_P g r_P}{m_P r_s r_P}}{\left\{\frac{[m_P r_P^2+I_P]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[(m_S+m_P)r_S^2+I_S]}\right\}}\sin z_2$$
$$- \frac{\frac{br^2}{m_P r_s r_P}}{\left\{\frac{[m_P r_P^2+I_P]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[(m_S+m_P)r_S^2+I_S]}\right\}} z_4 + \frac{\frac{\tau}{m_P r_s r_P}}{\left\{\frac{[m_P r_P^2+I_P]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[(m_S+m_P)r_S^2+I_S]}\right\}}$$
$$- \frac{\frac{\tau}{[(m_S+m_P)r_S^2+I_S]}}{\left\{\frac{[m_P r_P^2+I_P]}{m_P r_s r_P} - \frac{m_P r_s r_P}{[(m_S+m_P)r_S^2+I_S]}\right\}}$$

Linearizing the state-space expression and converting it into matrix form,

$$\dot{\boldsymbol{z}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{\frac{m_P g r_P}{[m_P r_P{}^2+I_P]}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} & \dfrac{\frac{br^2}{m_P r_S r_P}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} & -\dfrac{\frac{br^2}{[m_P r_P{}^2+I_P]}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} \\ 0 & -\dfrac{\frac{m_P g r_P}{m_P r_S r_P}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} & \dfrac{\frac{br^2}{[(m_S+m_P)r_S{}^2+I_S]}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} & -\dfrac{\frac{br^2}{m_P r_S r_P}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{\frac{1}{[m_P r_P{}^2+I_P]}-\frac{1}{m_P r_S r_P}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} \\ \dfrac{\frac{1}{m_P r_S r_P}-\frac{1}{[(m_S+m_P)r_S{}^2+I_S]}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} \end{bmatrix} \boldsymbol{\tau}$$

Hence, we have:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{\frac{m_P g r_P}{[m_P r_P{}^2+I_P]}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} & \dfrac{\frac{br^2}{m_P r_S r_P}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} & -\dfrac{\frac{br^2}{[m_P r_P{}^2+I_P]}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} \\ 0 & -\dfrac{\frac{m_P g r_P}{m_P r_S r_P}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} & \dfrac{\frac{br^2}{[(m_S+m_P)r_S{}^2+I_S]}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} & -\dfrac{\frac{br^2}{m_P r_S r_P}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\frac{1}{[m_P r_P{}^2+I_P]}-\frac{1}{m_P r_S r_P}}{\left\{\frac{[(m_S+m_P)r_S{}^2+I_S]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[m_P r_P{}^2+I_P]}\right\}} \\ \dfrac{\frac{1}{m_P r_S r_P}-\frac{1}{[(m_S+m_P)r_S{}^2+I_S]}}{\left\{\frac{[m_P r_P{}^2+I_P]}{m_P r_S r_P}-\frac{m_P r_S r_P}{[(m_S+m_P)r_S{}^2+I_S]}\right\}} \end{bmatrix}$$